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UNSTEADY SPREADING OF A THIN LAYER OF VISCOUS LIQUID OVER A CURVED SURFACE†

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The unsteady spreading of a thin layer of an incompressible viscous liquid over an impermeable curved surface, which occurs under the action of the gravity force, is considered. The solution of the boundary-value problem which arises reduces to solving a Cauchy problem for equations with a small number of independent variables. © 1998 Elsevier Science Ltd. All rights reserved.

1. The motion of the liquid is most conveniently considered in a special curvilinear orthogonal system of coordinates. The surface over which the liquid spreads is chosen as the initial coordinate surface. The y coordinate will be measured from it along the normal to it. On the initial coordinate surface, where $y = 0$, we will introduce curvilinear coordinates x, z such that the curves $x = \text{const}$ and $z = \text{const}$ form an orthogonal grid. At the point with coordinates x_0, z_0 this surface will define radii of curvature $R_x(x_0, z_0)$ and $R_z(x_0, z_0)$ of the coordinate lines $z = z_0$ and $x = x_0$, respectively [1]. The centres of normal curvature of these coordinate lines lie on the normal to the surface of inclination, and hence R_x and R_z are the values of the y coordinate for the corresponding centre of curvature.

Note that in order to use such a system of coordinates one needs to assume in addition that any of the centres of normal curvature lie outside the layer considered. Hence, when introducing the system of coordinates it is assumed that the layer is thin.

We will denote the Lamé coefficients for the coordinate lines $y = \text{const}, z = \text{const}$; $x = \text{const}, z = \text{const}$; and $x = \text{const}$ by l_x, l_y and l_z, respectively. In the case considered, they will be represented by the formulae

$$l_x = 1 + \frac{y}{R_x(x,z)}, \quad l_y = 1, \quad l_z = 1 + \frac{y}{R_z(x,z)} \tag{1.1}$$

We will denote the unit vectors of the coordinate trihedron at the point considered for the chosen system of coordinates by $\mathbf{o}_x, \mathbf{o}_y, \mathbf{o}_z$, the components of the velocity vector $\mathbf{v}$ by u, v, w, the components of the stress tensor by $t_{xx}, t_{yy}, t_{zz}, t_{xy}, t_{yz}, t_{xz}$, and the components of the unit vector of the acceleration due to gravity by $\gamma_x, \gamma_y, \gamma_z$. Note that the parameters $\gamma_x, \gamma_y, \gamma_z$ must be regarded as known functions of the coordinates x, y, z. For the chosen system of coordinates the position of the coordinate trihedron does not change along the coordinate line $x = \text{const}, z = \text{const}$ (i.e. along the normal to the surface of inclination), and hence the parameters $\gamma_x, \gamma_y, \gamma_z$ are independent of the y coordinate; in addition $\gamma_x^2 + \gamma_y^2 + \gamma_z^2 = 1$.

Using the well-known formulae and equations for orthogonal curvilinear coordinates, the equations of motion of an incompressible liquid in the chosen system of coordinates can be written as follows:

the equation of continuity

$$\frac{\partial}{\partial x}(l_y l_z u) + \frac{\partial}{\partial y}(l_z l_x v) + \frac{\partial}{\partial z}(l_x l_y w) = 0 \tag{1.2}$$

the equations of the change in angular momentum in projections on the directions of the unit vectors $\mathbf{o}_x$, $\mathbf{o}_y$ and $\mathbf{o}_z$

$$\rho\left(\frac{\partial u}{\partial t} + \frac{u}{l_x}\frac{\partial u}{\partial x} + \frac{v}{l_y}\frac{\partial u}{\partial y} + \frac{w}{l_z}\frac{\partial u}{\partial z} + \frac{uv}{l_x l_y}\frac{\partial l_x}{\partial y} + \frac{uw}{l_x l_z}\frac{\partial l_x}{\partial z} - \frac{v^2}{l_x l_y}\frac{\partial l_y}{\partial x} - \frac{w^2}{l_x l_z}\frac{\partial l_z}{\partial x}\right) = \tag{1.3}$$

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$$= g\rho\gamma_x + \frac{1}{l_x l_y l_z}\left(\frac{\partial}{\partial x}(l_y l_z t_{xx}) + \frac{\partial}{\partial y}(l_x l_z t_{xy}) + \frac{\partial}{\partial z}(l_y l_x l_{xz}) + \right.$$

$$\left. + l_z \frac{\partial l_x}{\partial y} t_{xy} + l_y \frac{\partial l_x}{\partial z} t_{xz} - l_z \frac{\partial l_y}{\partial x} t_{yy} - l_y \frac{\partial l_z}{\partial x} t_{zz}\right) \quad (xyz, uvw)$$

(only the first of these is presented, since the remaining two can be obtained by cyclic permutation of the notation x, y, z and u, v, w). Here t is the time and ρ is the density of the liquid.

The components of the strain rate tensor in a curvilinear orthogonal system of coordinates can be represented by the formulae

$$e_{xx} = 2\left(\frac{1}{l_x}\frac{\partial u}{\partial x} + \frac{v}{l_x l_y}\frac{\partial l_x}{\partial y} + \frac{w}{l_x l_z}\frac{\partial l_x}{\partial z}\right)$$

$$e_{xy} = \frac{l_x}{l_y}\frac{\partial}{\partial y}\frac{u}{l_x} + \frac{l_y}{l_x}\frac{\partial}{\partial x}\frac{v}{l_y} \quad (xyz, uvw) \tag{1.4}$$

(the unwritten equations are obtained by cyclic permutation).

We will consider the case when the stress tensors and the strain rate tensors are related by the linear equation

$$t_{ik} + p\delta_{ik} = \mu e_{ik} \quad (i, k = x, y, z) \tag{1.5}$$

Here

$$p = -\frac{1}{3}(t_{xx} + t_{yy} + t_{zz}), \quad \delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

and μ is the constant coefficient of the liquid viscosity.

Equations (1.2), (1.3) and (1.5), taking (1.4) into account, form a closed system of non-linear partial differential equations. Its solution must satisfy the boundary conditions on the surface over which the flow occurs and on the free surface of the liquid. We will give them in orthogonal curvilinear coordinates. On the surface over which the flow occurs the following obvious condition must be satisfied:

$$y = 0: \; v = 0 \tag{1.6}$$

In some cases we can assume (following Stokes) the no-slip condition

$$y = 0: \; u = 0, \; w = 0 \tag{1.7}$$

But a case may arise when the slip condition must be specified on the surface over which the flow occurs. We will denote the shear stress vector on the surface over which flow occurs by t_s, and the normal pressure p_n by $p_n = -t_{yy}$, $t_s = \sqrt{(t^2_{xy} + t^2_{zy})}$. One of the possible assumptions is the one made by Newton, that the shear stress on the surface is proportional to the rate of spread of the liquid on it

$$y = 0: \; \mathbf{t}_s = \rho c_* \mathbf{v} \tag{1.8}$$

Here c_* is a material constant representing the spreading of the medium over the surface. Condition (1.8) is equivalent to the two scalar conditions

$$y = 0: \; \rho c_* u = t_{xy}, \; \rho c_* w = t_{yz} \tag{1.9}$$

In some other cases the nature of the interaction between the moving medium and the surface requires us to assume a constant shear stress on it [2]

$$y = 0: \; t_{xy}^2 + t_{zy}^2 = \tau_*^2 \tag{1.10}$$

Here τ_* is a specified physical constant. In the case of spatial motion it is natural to assume that

$$y=0:\ t_{xy}=\tau_*\frac{u}{q},\ t_{zy}=\tau_*\frac{w}{q},\quad q=\sqrt{u^2+w^2} \tag{1.11}$$

On the free surface of the liquid $y = h(x, z, t)$ the kinematic condition

$$y=h(x,z,t):\ \frac{\partial h}{\partial t}=v-\frac{u}{l_x}\frac{\partial h}{\partial x}-\frac{w}{l_z}\frac{\partial h}{\partial z} \tag{1.12}$$

is specified. This equation, together with the initial condition

$$t=0:\ h(x,z,0)=h^{(0)}(x,z) \tag{1.13}$$

also serves to define the free surface.

Henceforth, we will use the notation u_1, v_1 and w_1 to denote the components of the velocity vector of particles on the free surface.

The boundary conditions on the free surface reduce to the following: the shear stresses on it vanish, while the normal stress is equal to the atmospheric pressure p_a with opposite sign

$$y=h(x,z,t):\ t_{sx}=t_{sy}=t_{sz}=0,\ t_n=-p_a$$

Here t_n is the value of the vector $\mathbf{t}_n$ of the normal stress on the free surface, and t_{sx}, t_{sy} and t_{sz} are the components of the vector $\mathbf{t}_s$ of the shear stresses on the free surface.

We will denote the total stress vector on the free surface by $\mathbf{t}$, with components t_x, t_y and t_z, and the unit vector of the normal to the free surface by $\mathbf{n}$, with the components n_x, n_y and n_z. We have

$$t_{sx}=t_x-t_nn_x,\ t_{sy}=t_y-t_nn_y,\ t_{sz}=t_z-t_nn_z$$

The boundary conditions for the stresses on the free surface can be represented in the form

$$y=h(x,z,t):\ (t_{xx}+p_a)n_x+t_{xy}n_y+t_{xz}n_z=0\quad (xyz) \tag{1.14}$$

(two further conditions are obtained by cyclic permutation of the subscripts).

Equation (1.12) can be converted to a form in which it is used in practice for the numerical solution of the problem of the motion of the medium. To do this we integrate the equation of continuity (1.2) over y from 0 to $h(x, z, t)$ and we obtain

$$v_1=-\frac{1}{l_xl_z}\int_0^h\left(\frac{\partial(l_zu)}{\partial x}+\frac{\partial(l_xw)}{\partial z}\right)dy$$

Since

$$\int_0^h\frac{\partial(l_zu)}{\partial x}dy=\frac{\partial}{\partial x}\int_0^h l_zudy-l_zu_1\frac{\partial h}{\partial x}\quad (xz,uw)$$

we can represent Eq. (1.12) in the form

$$\frac{\partial h}{\partial t}=-\frac{1}{l_xl_z}\left(\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z}\right) \tag{1.15}$$

$$\left(U(x,z,t)=\int_0^h l_zudy,\ W(x,z,t)=\int_0^h l_xwdy\right)$$

The solution of the problem must also satisfy the initial conditions, which can be represented in the form

$$t=0:\ h=h^{(0)}(x,z),\ u=u^{(0)}(x,z),\ w=w^{(0)}(x,z) \tag{1.16}$$

where $h^{(0)}(x, z)$, $u^{(0)}(x, z)$, $w^{(0)}(x, z)$ are specified functions.

2. The system of equations (1.2)–(1.5) is extremely difficult to solve in general form. But when the thickness of the spreading layer is small compared with its extent on the surface, the curvature of which has no very sharp changes (the functions $h(x, z, t)$, $R_x(x, z)$ and $R_z(x, z)$ must be continuous and piecewise-smooth), the equations of motion of the liquid can be greatly simplified, and this enables us to advance considerably when solving the problem. A certain analogy with boundary-layer theory occurs here for the high-speed flow of a viscous liquid around a surface but in the case considered the thickness of the layer is finite and is determined quite accurately when solving the problem.

To investigate the possibility of the above simplifications, we will change in Eqs (1.2)–(1.5) and boundary conditions (1.6)–(1.14) to dimensionless variables using the formulae

$$\begin{gathered}
x = L_*\xi,\ \ z = L_*\zeta,\ \ y = h_*\eta,\ \ u = u_*\bar{u},\ \ v = v_*\bar{v},\ \ w = u_*\bar{w} \\
t = \frac{L_*}{u_*}\tau,\ \ R_x = L_*\rho_1(\xi,\zeta),\ \ R_z = L_*\rho_3(\xi,\zeta) \\
l_x = l_1 = 1 + \frac{h_*}{L_*}\frac{\eta}{\rho_1},\ \ l_y = l_1 = 1,\ \ l_z = l_3 = 1 + \frac{h_*}{L_*}\frac{\eta}{\rho_3} \\
h_0(x,z) = h_*\bar{h}_0(\xi,\zeta),\ \ h(x,z,t) = h_*\bar{h}(\xi,\zeta,\tau) \\
p = \rho g h_*\psi,\ \ p_a = \rho g h_*\bar{p}_a,\ \ t_{xx} = \rho g h_*\tau_{11},\ \ t_{yy} = \rho g h_*\tau_{22},\ \ t_{zz} = \rho g h_*\tau_{33} \\
t_{xy} = \rho g h_*\tau_{12},\ \ t_{xz} = \rho g h_*\tau_{13},\ \ t_{yz} = \rho g h_*\tau_{23},\ \ s = \rho g h_*\sigma \\
\sigma = \left(\frac{1}{2}(\psi+\tau_{11})^2 + (\psi+\tau_{22})^2 + (\psi+\tau_{33})^2 + 2(\tau_{12}^2+\tau_{13}^2+\tau_{23}^2)\right)^{1/2}
\end{gathered} \tag{2.1}$$

Here L_* is the characteristic scale of the dimensions of the layer along the surface, h_* is its characteristic thickness, and u_* and υ_* are the characteristic values for the components of the velocity tangential and normal to the surface for the flow considered, respectively.

The quantities L^*, h_*, μ are known quantities from the formulation of the problem, while the quantities u_* and υ_* are unknown in advance, and there is some arbitrariness in choosing them. We will consider flows for which the parameter $\varepsilon = h_*/L_*$ can be assumed to be a small quantity compared with unity.

The simplifications, connected with the thinness of the spreading layer, essentially consist of the fact that in the equations, represented in dimensionless form, we will neglect terms which vanish as $\varepsilon \to 0$.

The Lamé parameters can be represented in the form

$$l_x = l_1 = 1 + \varepsilon\frac{\eta}{\rho_1},\ \ l_z = l_3 = 1 + \varepsilon\frac{\eta}{\rho_3},\ \ l_y = l_2 = 1$$

Apart from quantities $O(\varepsilon^2)$, we have the equations

$$l_1 l_3 = 1 + \varepsilon\left(\frac{1}{\rho_1} + \frac{1}{\rho_3}\right),\ \ \frac{1}{l_1} = \left(1 + \varepsilon\frac{\eta}{\rho_1}\right)^{-1} = 1 - \varepsilon\frac{\eta}{\rho_1}$$

etc.

Since $\upsilon = 0$ when $y = 0$, we can conclude from a consideration of the equation of continuity (1.2) that we must take $\upsilon_* = \varepsilon u_*$. The equation of continuity itself, in dimensionless form, if we neglect terms $O(\varepsilon)$, reduces to the form

$$\frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \bar{v}}{\partial \eta} + \frac{\partial \bar{w}}{\partial \zeta} = 0 \tag{2.2}$$

The components of the deformation rate tensor, apart from terms $O(\varepsilon)^2$, can be reduced to the form

$$e_{xx} = 2\varepsilon\frac{u_*}{h_*}\frac{\partial\bar{u}}{\partial\xi},\quad e_{yy} = 2\varepsilon\frac{u_*}{h_*}\frac{\partial\bar{v}}{\partial\eta},\quad e_{zz} = 2\varepsilon\frac{u_*}{h_*}\frac{\partial\bar{w}}{\partial\zeta}$$

$$e_{xy} = \frac{u_*}{h_*}\left(\frac{\partial\bar{u}}{\partial\eta} - \varepsilon\frac{\bar{u}}{\rho_1}\right),\quad e_{zy} = \frac{u_*}{h_*}\left(\frac{\partial\bar{w}}{\partial\eta} - \varepsilon\frac{\bar{w}}{\rho_3}\right),\quad e_{xz} = \varepsilon\frac{u_*}{h_*}\left(\frac{\partial\bar{w}}{\partial\xi} + \frac{\partial\bar{u}}{\partial\zeta}\right)$$

Representing (1.5) in dimensionless variables, we obtain

$$\begin{gathered}\tau_{11} = -\psi + \Lambda\varepsilon\hat{\tau}_{11},\quad \tau_{22} = -\psi + \Lambda\varepsilon\hat{\tau}_{22},\quad \tau_{33} = -\psi + \Lambda\varepsilon\hat{\tau}_{33}\\ \tau_{12} = \Lambda\hat{\tau}_{12},\quad \tau_{23} = \Lambda\hat{\tau}_{23},\quad \tau_{13} = \varepsilon\Lambda\hat{\tau}_{13}\\ \hat{\tau}_{12} = \frac{\partial\bar{u}}{\partial\eta} - \varepsilon\frac{\bar{u}}{\rho_1},\quad \hat{\tau}_{23} = \frac{\partial\bar{w}}{\partial\eta} - \varepsilon\frac{\bar{w}}{\rho_3},\quad \hat{\tau}_{13} = \frac{\partial\bar{u}}{\partial\zeta} + \frac{\partial\bar{w}}{\partial\xi}\\ \hat{\tau}_{11} = 2\frac{\partial\bar{u}}{\partial\xi},\quad \hat{\tau}_{22} = 2\frac{\partial\bar{v}}{\partial\eta},\quad \hat{\tau}_{33} = 2\frac{\partial\bar{w}}{\partial\zeta}\quad \left(\Lambda = \frac{\mu u_*}{\rho g h_*^2}\right)\end{gathered} \tag{2.3}$$

Using (2.3), apart from quantities $O(\varepsilon)$, the system of equations (1.3) can be represented in the form

$$\begin{gathered}F\left(\frac{\partial\bar{u}}{\partial\tau} + \bar{u}\frac{\partial\bar{u}}{\partial\xi} + \bar{v}\frac{\partial\bar{u}}{\partial\eta} + \bar{w}\frac{\partial\bar{u}}{\partial\zeta}\right) - \gamma_1(\xi,\zeta) - \Lambda\frac{\partial^2\bar{u}}{\partial\eta^2} = 0\\ F\left(\frac{\partial\bar{w}}{\partial\tau} + \bar{u}\frac{\partial\bar{w}}{\partial\xi} + \bar{v}\frac{\partial\bar{w}}{\partial\eta} + \bar{w}\frac{\partial\bar{w}}{\partial\zeta}\right) - \gamma_3(\xi,\zeta) - \Lambda\frac{\partial^2\bar{w}}{\partial\eta^2} = 0\\ F\left(\frac{\bar{u}^2}{\rho_1} + \frac{\bar{w}^2}{\rho_3}\right) + \gamma_2(\xi,\zeta) - \frac{\partial\bar{p}}{\partial\eta} = 0\quad \left(F = \frac{u_*^2}{gL_*} = \varepsilon\frac{u_*^2}{gh_*}\right)\end{gathered} \tag{2.4}$$

For a thin layer, apart from quantities $O(\varepsilon)$, we have $\sigma = \sqrt{(\tau_{12}^2 + \tau_{23}^2)}$.

We will transform the boundary conditions. Assuming that the direction cosines of the normal to the free surface, apart from terms $O(\varepsilon^2)$, can be represented in the form

$$\nu_x = -\varepsilon\frac{\partial\bar{h}}{\partial\xi},\quad \nu_y = 1,\quad \nu_z = -\varepsilon\frac{\partial\bar{h}}{\partial\zeta}$$

and bearing (2.3) in mind, boundary conditions (1.14) on the free surface can be written in the following form in dimensionless variables with the same accuracy

$$\eta = \bar{h}(\xi,\zeta,\tau):\ \bar{p}_a = \Psi - \Lambda\varepsilon\left(\tau_{22} + \tau_{12}\frac{\partial\bar{h}}{\partial\xi} + \tau_{23}\frac{\partial\bar{h}}{\partial\zeta}\right)$$

$$\tau_{12} - \varepsilon(\bar{p}_a - \psi)\frac{\partial\bar{h}}{\partial\xi} = 0,\quad \tau_{23} - \varepsilon(\bar{p}_a - \psi)\frac{\partial\bar{h}}{\partial\zeta} = 0$$

It follows from these equations that, apart from quantities $O(\varepsilon)$

$$\eta = \bar{h}(\xi,\zeta,\tau):\ \tau_{12} = \tau_{23} = 0,\quad \psi = \bar{p}_a \tag{2.5}$$

We will write condition (1.12) in dimensionless form, apart from quantities $O(\varepsilon)$

$$\eta = \bar{h}(\xi,\zeta,\tau):\ \bar{v} = \frac{\partial\bar{h}}{\partial\tau} + \bar{u}\frac{\partial\bar{h}}{\partial\xi} + \bar{w}\frac{\partial\bar{h}}{\partial\zeta} \tag{2.6}$$

Equation (1.15) can be represented in the following form in dimensionless variables with the same accuracy

$$\frac{\partial \bar{h}}{\partial \tau}+\frac{\partial \bar{U}}{\partial \xi}+\frac{\partial \bar{W}}{\partial \zeta}=0,\quad \bar{U}=\int_0^{\bar{h}} \bar{u}\,d\eta,\quad \bar{W}=\int_0^{\bar{h}} \bar{w}\,d\eta \tag{2.7}$$

No difficulties arise in writing the boundary conditions (1.6)–(1.10) on the surface in dimensionless form.

The equations obtained by taking the limit as $\varepsilon \to 0$ form a system of equations of a "thin" layer, which also serves as the object of our further consideration. The dimensionless parameters F and Λ may have different values depending on the particular features of the problem considered.

The case when $\Lambda \simeq 1$ and $F \ll 1$ corresponds to slow ("creep") flows and has been investigated fairly fully. The equations which have been obtained can be used, for example, to construct a theory of a lubricating layer [3, 4], to describe the motion of isothermal glaciers [5] and landslides.

The case when $F \simeq 1$ is the most common, where the smallness of the parameter Λ, even if occurs, does not lead to any further simplifications of the equations obtained, since in these equations it serves as the coefficient of the higher derivatives of the unknown functions.

We will now return to the initial notation of the variables and constant parameters of the problem. The system of simplified equations can then be represented in the form

$$\begin{gathered}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\\
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-g\gamma_x(x,z)-\nu\frac{\partial^2 u}{\partial y^2}=0\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-g\gamma_z(x,z)-\nu\frac{\partial^2 w}{\partial y^2}=0\\
\frac{u^2}{R_1}+\frac{w^2}{R_3}+g\gamma_y(x,z)-\frac{\partial p}{\partial y}=0
\end{gathered} \tag{2.8}$$

Here $\nu = \mu/\rho$ is the kinematic coefficient of viscosity.

The boundary conditions on the free surface can be written in the form

$$y=h(x,z,t):\ t_{xy}=t_{yz}=0,\ v=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+w\frac{\partial h}{\partial z},\ p=p_a \tag{2.9}$$

Equation (1.15) can be represented in the form

$$\frac{\partial h}{\partial t}+\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z}=0,\quad U=\int_0^h u\,dy,\quad W=\int_0^h w\,dy \tag{2.10}$$

The form of the boundary conditions (1.6)–(1.10) on the surface and the initial conditions (1.16) do not contain any changes. It is more convenient to represent the boundary conditions on the surface in the following form, using Newton's hypothesis

$$y=0:\ v=0,\ u=l\frac{\partial u}{\partial y},\ w=l\frac{\partial w}{\partial y}\quad \left(l=\frac{\nu}{c_*}\right)$$

Note that the system of four equations (2.8) obtained and boundary conditions (2.9) and (1.6)–(1.10) splits into a closed system *of the first three equations of (2.8) and boundary conditions (2.9), excluding the last one, condition (1.16) and one of the conditions (1.7), (1.9) or (1.11),* and the last of Eqs (2.8) with the last of boundary conditions (2.9), the solution of which can be constructed after the solution of the system, distinguished by italics, has been obtained.

Henceforth, to obtain the equations we will consider the case corresponding to the plane problem, when the dependent variables are independent of the z coordinate and their number is reduced due to the fact that the quantities w and t_{yz} identically vanish. However, this limitation is unimportant, since the main results obtained for the plane problem can be extended without particular difficulty to the three-dimensional case.

3. For the plane problem the spreading equations can be represented in the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}=g\alpha(x)+\nu\frac{\partial^2 u}{\partial y^2} \tag{3.1}$$

Here $\alpha(x)$ is a specified function, which is defined by the angle between the surface over which the flow occurs and the horizontal plane.

To solve system (3.1) the physical and kinematic boundary conditions on the free surface and on the surface around which flows occurs—the limits of the flow, must be satisfied. We will henceforth arrange to distinguish the values of the flow parameters at points of its boundaries using subscripts: a one on the free surface and a zero on the surface over which flow occurs. The boundary conditions on the free surface will then be written in the form

$$\left(\frac{\partial u}{\partial y}\right)_1=0,\quad v_1=\frac{\partial h}{\partial t}+u_1\frac{\partial h}{\partial x} \tag{3.2}$$

The boundary conditions on the surface over which the flow occurs will be represented in the form, using Newton's hypothesis

$$v_0=0,\quad u_0=l\left(\frac{\partial u}{\partial y}\right)_0\quad \left(l=\frac{\mu}{\rho c_*}\right) \tag{3.3}$$

(l is a physical constant with the dimension of length).

Using the no-slip condition, the boundary conditions have the form

$$v_0=0,\ u_0=0 \tag{3.4}$$

Assuming the shear stress on the surface over which flow occurs is constant, the boundary condition will have the form

$$v_0=0,\quad \left(\frac{\partial u}{\partial y}\right)_0=\omega_*\quad \left(\omega_*=\frac{\tau_*}{\mu}\right) \tag{3.5}$$

Below we give a detailed description of a method of constructing solutions of these equations, the basic ideas of which were outlined earlier [6], and also presented at the conference "Modern Problems of Mathematics and Mechanics", devoted to 175 years of P. L. Chebyshev (Moscow State University, 14–19 May 1996); the content of this paper was published in [7].

Integrating Eqs (3.1) over the layer thickness, we obtain integral conditions which the required solution must satisfy

$$\frac{\partial h}{\partial t}+\frac{\partial U}{\partial x}=0,\quad \frac{\partial U}{\partial t}=hg\alpha(x)-\frac{\partial U_1}{\partial x}-\nu\left(\frac{\partial u}{\partial y}\right)_0 \tag{3.6}$$

$$\left(U=\int_0^h u\,dy,\quad U_1=\int_0^h u^2dy\right)$$

Multiplying the first equation of (3.1) by u and integrating over the y coordinate within the limits $[0, h]$, we obtain another integral condition

$$\frac{\partial U_1}{\partial t}=2g\alpha(x)U-\frac{\partial U_2}{\partial x}-2\nu\Omega-2\nu u_0\left(\frac{\partial u}{\partial y}\right)_0 \tag{3.7}$$

$$\left(U_2=\int_0^h u^3dy,\quad \Omega=\int_0^h\left(\frac{\partial u}{\partial y}\right)^2 dy\right)$$

We will seek a solution of system (3.1) by representing the longitudinal velocity $u(t, x, y)$ in the form of a sum

$$u = b_\gamma(t,x)g_\gamma(\eta) + a_\gamma(t,x)f_\gamma(\eta) \tag{3.8}$$

(summation is carried out over the subscript γ, $\gamma = 0, 1, \dots, N$; $N > 2$). Here

$$\eta = \frac{y}{h},\quad a_0 = u_0,\quad a_1 = h\left(\frac{\partial u}{\partial y}\right)_0,\quad a_2 = h^2\left(\frac{\partial^2 u}{\partial y^2}\right)_0,\dots, a_N = h^N\left(\frac{\partial^N u}{\partial y^N}\right)_0$$

$$b_0 = u_1,\quad b_1 = h\left(\frac{\partial u}{\partial y}\right)_1,\quad b_2 = h^2\left(\frac{\partial^2 u}{\partial y^2}\right)_1,\dots, b_N = h^N\left(\frac{\partial^N u}{\partial y^N}\right)_1$$

$g_i(\eta)$ and $f_i(\eta)$ ($i = 0, 1, 2, \dots, N$) are specified functions of the sole variable η, which satisfy the following conditions on the edges of the region $0 < y < h(t, x)$

$$f_{i,i} = g_{i,i} = 1;\quad f_{i,k} = g_{i,k} = 0\ \ (k \neq i),\quad \varphi_{i,k} = \gamma_{i,k} = 0 \quad (0 \leqslant i,k \leqslant N) \tag{3.9}$$

We have used the following notation here

$$f_{i,k} = f_i^{(k)}(0),\ \ \varphi_{i,k} = f_i^{(k)}(1)\ \ (f_{i,0} = f_i(0),\ \ \varphi_{i,0} = f_i(1))$$

$$g_{i,k} = g_i^{(k)}(1),\ \ \gamma_{1,k} = g_1^{(k)}(0)\ \ (g_{1,0} = g_1(1),\ \ \gamma_{1,0} = g_1(0))$$

(note that $(\partial u/\partial y)_1 = 0$, $(\partial^3 u/\partial y)_1$, and hence we can take $g_1(\eta) \equiv 0, g_3(\eta) \equiv 0$).

When the boundary condition on the surface over which flow occurs is specified in the form (3.3) or (3.5) (the first case), the following conditions are assumed to be satisfied

$$A_0 = \int_0^1 f_0(\eta)d\eta = O\left(\frac{1}{2}\right),\quad A_1 = \int_0^1 g_0(\eta)d\eta = O\left(\frac{1}{2}\right)$$

for the functions $f_0(\eta)$ and $g_0(\eta)$ (the notation $A = O(1/2)$ denotes that, for A which satisfies the conditions $0 < A < 1$, the relation $A \ll 1$ is untrue), and when it is specified in the form (3.4) (the second case), the following conditions are assumed to be satisfied

$$A_0 = \int_0^h f_1(\eta)d\eta = O\left(\frac{1}{2}\right),\quad A_1 = \int_0^h g_0(\eta)d\eta = O\left(\frac{1}{2}\right)$$

for the functions $f_1(\eta)$ and $g_0(\eta)$ (note that $a_0 \equiv 0$, and hence we can put $f_0 \equiv 0$ in this case).

We also require that the following conditions must be satisfied for the functions $f_i(\eta)$ and $g_i(\eta)$ ($i > 1$)

$$\int_0^1 |f_i(\eta)|d\eta \ll 1,\quad \int_0^1 |g_i(\eta)|d\eta \ll 1 \tag{3.10}$$

in the first of these cases, and the conditions

$$\int_0^1 |f_{i+1}(\eta)|d\eta \ll 1,\quad \int_0^1 |g_i(\eta)|d\eta \ll 1 \tag{3.11}$$

in the second of these cases (in terms of [7], the corresponding functions $f_i(\eta)$ and $g_i(\eta)$ form a set of functions of small filling).

It can be shown that for functions which satisfy conditions (3.9), conditions (3.10) or (3.11) will be satisfied automatically for sufficiently high values of the parameter N.

Consider the first case.

The functionals $U\{t, x\}$, $U_1\{t, x\}$, $U_2\{t, x\}$ and $\Omega_1\{t, x\}$ can be represented with high accuracy in the following form (we ignore integrals of functions of small filling)

$$
\begin{aligned}
&U = h(A_0u_0 + A_1u_1)\\
&U_1 = h(A_{00}u_0^2 + 2A_{01}u_0u_1 + A_{11}u_1^2)\\
&U_2 = h(A_{000}u_0^3 + 3A_{001}u_0^2u_1 + 3A_{011}u_0u_1^2 + A_{111}u_1^3)\\
&\Omega = (B_{00}u_0^2 + 2B_{01}u_0u_1 + B_{11}u_1^2)/h
\end{aligned}
\tag{3.12}
$$

Here

$$
A_0 = \int_0^h f_0(\eta)d\eta, \quad A_1 = \int_0^h g_0(\eta)d\eta
$$

$$
A_{00} = \int_0^1 f_0^2(\eta)d\eta, \quad A_{01} = \int_0^1 f_0(\eta)g_0(\eta)d\eta, \quad A_{11} = \int_0^1 g_0^2(\eta)d\eta
$$

$$
A_{000} = \int_0^1 f_0^3(\eta)d\eta, \quad A_{001} = \int_0^1 f_0^2(\eta)g_0(\eta)d\eta
$$

$$
A_{011} = \int_0^1 f_0(\eta)g_0^2(\eta)d\eta, \quad A_{111} = \int_0^1 g_0^3(\eta)d\eta
$$

$$
B_{00} = \int_0^1 (f_0'(\eta))^2 d\eta, \quad B_{01} = \int_0^1 f_0'(\eta)g_1'(\eta)d\eta, \quad B_{11} = \int_0^1 (g_1'(\eta))^2 d\eta
$$

Using (3.12), we can represent Eqs (3.6) and (3.7) in the form

$$
\begin{aligned}
&\frac{\partial h}{\partial t} = -\frac{\partial U}{\partial h}\frac{\partial h}{\partial x} - \frac{\partial U}{\partial u_0}\frac{\partial u_0}{\partial x} - \frac{\partial U}{\partial u_1}\frac{\partial u_1}{\partial x}\\
&\frac{\partial U}{\partial u_0}\frac{\partial u_0}{\partial t} + \frac{\partial U}{\partial u_1}\frac{\partial u_1}{\partial t} = \mathcal{C}_0\frac{\partial h}{\partial x} + \mathcal{A}_0\frac{\partial u_0}{\partial x} + \mathcal{B}_0\frac{\partial u_1}{\partial x} + gh\alpha(x) - \nu\left(\frac{\partial u}{\partial y}\right)_0\\
&\frac{\partial U_1}{\partial u_0}\frac{\partial u_0}{\partial t} + \frac{\partial U_1}{\partial u_1}\frac{\partial u_1}{\partial t} = \mathcal{C}_1\frac{\partial h}{\partial x} + \mathcal{A}_1\frac{\partial u_0}{\partial x} + \mathcal{B}_1\frac{\partial u_1}{\partial x} + 2g\alpha(x)U - 2\nu\Omega - 2\frac{\nu}{l}u_0^2
\end{aligned}
\tag{3.13}
$$

Here

$$
\mathcal{C}_0 = \frac{\partial U}{\partial h}\frac{\partial U}{\partial h} - \frac{\partial U_1}{\partial h}, \quad \mathcal{A}_0 = \frac{\partial U}{\partial h}\frac{\partial U}{\partial u_0} - \frac{\partial U_1}{\partial u_0}, \quad \mathcal{B}_0 = \frac{\partial U}{\partial h}\frac{\partial U}{\partial u_1} - \frac{\partial U_1}{\partial u_1}
$$

$$
\mathcal{C}_1 = \frac{\partial U_1}{\partial h}\frac{\partial U}{\partial h} - \frac{\partial U_2}{\partial h}, \quad \mathcal{A}_1 = \frac{\partial U_1}{\partial h}\frac{\partial U}{\partial u_0} - \frac{\partial U_2}{\partial u_0}, \quad \mathcal{B}_1 = \frac{\partial U_1}{\partial h}\frac{\partial U}{\partial u_1} - \frac{\partial U_2}{\partial u_1}
$$

Equations (3.13) (taking into account the fact that in the case considered either $(\partial u/\partial y)_0 = u_0/l$ or $(\partial u/\partial y)_0 = \omega_*$) form a system which is closed with respect to the unknown functions $h(t,x)$, $u_0(t,x)$, $u_1(t,x)$. Here the last two equations of (3.13) can always be solved for the derivatives $\partial u0/\partial t$ and $\partial u1/\partial t$. This system is of the hyperbolic type, but the directions t = const are not characteristic directions for it (it is curious that for the initial system (3.1), which is of the parabolic type, the directions t = const are characteristic directions), and hence the initial conditions

$$
t = 0{:}\ \ h = h^{(0)}(x), \ \ u_0 = u^{(0)}(x), \ \ u_1 = u^{(0)}(x)
\tag{3.14}
$$

where $h^{(0)}(x)$, $u^{(0)}(x)$ are specified functions, form a Cauchy problem for this system, methods of solving which are well developed.

When the functions $h(t,x)$, $u_0(t,x)$, $u_1(t,x)$ have been obtained, Eqs (3.1) enable one, taking the boundary conditions into account, to determine $(\partial^i u/\partial y^i)_0$, $(\partial^i u/\partial y^i)_1$ for $i > 1$, and formula (3.8) then gives the required solution of the problem.

The following question arises: to what extent is the solution of system (3.1), for which the longitudinal velocity $u(t, x, y)$ is represented by the sum (3.5), exact?

Note that for representation (3.5), Eqs (3.1) are satisfied with multiplicity $N - 1$ on the free surface and on the surface over which flow occurs. It is obvious that the accuracy of the solution will increase as N increases. This observation cannot, of course, serve as a satisfactory answer to the question, but the method considered contains the constructive possibility for solving it.

When the boundary condition on the surface over which flow occurs is specified in the form (3.5), the integral conditions (3.6) and (3.7) can be represented in the form

$$\frac{\partial h}{\partial t}+\frac{\partial U}{\partial x}=0,\ \frac{\partial U}{\partial t}=hg\alpha(x)-\frac{\partial U_1}{\partial x}-\nu\left(\frac{\partial u}{\partial y}\right)_0,\ \frac{\partial U_1}{\partial t}=2g\alpha(x)U-\frac{\partial U_2}{\partial x}-2\nu\Omega \tag{3.15}$$

while the functionals can be represented in the form (3.12), except that we must replace u_0 by a_1 in (3.12), and when calculating the coefficients $A_0, A_{00}, A_{01}, A_{000}, A_{001}, A_{011}, B_{00}, B_{01}$ the function $f_0(\eta)$ must be replaced by $f_1(\eta)$.

To determine the unknown functions $h(t,x), a_1(t,x), u_1(t,x)$ a system of equations arises which differs from (3.13) by the global replacement of u_0 by a_1, replacement of the last term on the right-hand side of the second equation by $-\nu a_1/h$ and the absence of the last term on the right-hand of the third equation.

The initial conditions can be represented in the form

$$t=0:\ h=h^{(0)}(x),\ a_1=h^{(0)}(x)\omega^{(0)}(x),\ u_1=u_1^{(0)}(x)$$

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